THREE RESULTS OF COMBINATORIAL GAME TOADS AND FROGS

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Abstract. We show values of the starting positions $\mathrm{T}^a\square\square\square\mathrm{F}^b$, $\mathrm{T}^a\square\square\mathrm{F}^b$ and $\mathrm{T}^a\square\square\square\mathrm{FFF}$. The last two positions are Erickson's conjectures.

1. Introduction:

The game *Toads and Frogs*, invented by Richard Guy, is extensively discussed in "Winning Ways"[1], the famous classic by Elwyn Berlekemp, John Conway, and Richard Guy, that is the *bible* of combinatorial game theory.

This game got so much coverage because of the simplicity and elegance of its rules, the beauty of its analysis, and as an example of a combinatorial game whose positions do not always have values that are numbers.

Rule

The game is played on a $1 \times n$ strip with either Toad(T) , Frog(F) or $\square$ on the squares. Left plays T and Right plays F. T may move to the immediate square on its right, if it happens to be empty, and F moves to the next empty square on the left, if it is empty. If T and F are next to each other, they have an option to jump over one another, in their designated directions, provided they land on an empty square. (see [1] page 14).

In symbols: the following moves are legal for Toad:

$\ldots\mathrm{T}\square\ldots \quad\rightarrow\quad \ldots\square\mathrm{T}\ldots,$
$\ldots\mathrm{TF}\square\ldots \quad\rightarrow\quad \ldots\square\mathrm{FT}\ldots \quad ,$

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and the following moves are legal for Frog:

$\ldots\Box\mathrm{F}\ldots \quad\rightarrow\quad \ldots\mathrm{F}\Box\ldots,$
$\ldots\Box\mathrm{TF}\ldots \quad\rightarrow\quad \ldots\mathrm{FT}\Box\ldots.$

Two players take turn moving their pieces. The first player who runs out of move loses.

To be able to understand the present article, reader need a minimum knowledge of combinatorial game theory, that can be found in [1]. In particular, readers should be familiar with the notions of *value* of a game and sum games which are the bypass reversible move rule, dominated options rule and inequality of two games. (see [1] page 33, 62-64).

The only notations we use are $*(=\{0 \mid 0\})$ and $n*(=\{n \mid n\})$. We will not use any shorthand notation like $\uparrow$, $\Uparrow$, etc.

Background Story

Already in "Winning Ways"[1] there is some analysis of Toads and Frogs positions, but on *specific*, small boards, such as TTT□FF. In 1996, Erickson[2] analyzed more general positions such as $\mathrm{T}\Box^a\mathrm{F}$ for any a. At the end he made five conjectures about the values of some families of positions. All of them are starting positions (positions where all T are leftmost and all F are rightmost).

In [3], Doron Zeilberger and the author discussed the new algorithm called *the symbolic finite-state method*. This method is an automated algorithm to prove the values of all positions in the given class of this game all at the same time. For example values of any positions with one □, one F and any number of T was shown and proved using this method. In practice, we knows all the positions of the class □F, □□F, □□□F, □□□□F, □□□□□F, □FF, □□FF, □□□FF, □FFF, TF. All the results are in [5].

Unfortunately, the symbolic finite-state method does not work with the positions where there are variables on both T and F or both □ and F such as $\mathrm{T}^a\Box\Box\mathrm{F}^b$. However some of these positions have nice values and are of great interests. In [4], the author analyzed some of these positions. In particular, we show the value of $\mathrm{T}^a\Box\Box\mathrm{F}^a$ is an infinitesimal for $a \geq 4$ and the value of all positions with one □.

Current Work

The values of the starting positions are of a great interest. Some of them have been investigated in Erickson's paper [2]. The starting position with the variables on both T and F interested the author a lot. In this paper, we show the values of the three starting positions namely $\mathrm{T}^a\square\square\square\mathrm{F}^b$, $\mathrm{T}^a\square\square\mathrm{F}^b$ and $\mathrm{T}^a\square\square\square\mathrm{FFF}$. All these three positions have beautiful values. The proofs however seem tedious. The techniques used to prove these positions are similar. The author decides to move the proofs of $\mathrm{T}^a\square\square\mathrm{F}^b$ and $\mathrm{T}^a\square\square\square\mathrm{FFF}$ to the appendices to make the paper more readable.

In section 3, we show the value of $\mathrm{T}^a\square\square\square\mathrm{T}^b, a \geq 4, b \geq 4$. These positions are important starting positions which cannot be proved by the symbolic finite-state method. The proof is the shortest amongst the three positions.

In the appendix A, we discuss the values of the position $\mathrm{T}^a\square\square\mathrm{F}^b, a > b \geq 2$. The value of this position is the first conjecture of Jeff Erickson in [2]. The proof is not long but tricky. The values of the positions $\mathrm{T}^a\square\square\square\square\mathrm{T}^b, a \geq 6, b \geq 6$ and $\mathrm{T}^a\square\square\square\square\square\mathrm{T}^b$ are still unknown.

In the appendix B, we show the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}, a \geq 5$. In theory, we can apply the symbolic finite-state method to prove the values of all positions in the class $\square\square\square\mathrm{FFF}$. The result above follows as one of the special case of many positions in this class. But as we mentioned in [4], we could not get the computer to conjecture all the values of all the positions in this class yet. And it would takes days for human to do conjectures by hand. For now we prove the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}$, $a \geq 5$, which is Jeff Erickson's conjecture 3, by hand. The proof is however assisted by computer program the author wrote in Maple.

We do not have an automated algorithm for position with variable on both T and F yet. The proofs are assisted by the old program in making conjectures and doing computation. In the future, we hope to have a new method (hopefully along the same line as the symbolic finite-state method) or computer program to make the proofs more automatic or at least shorten them.

2. Convention and Lemma.

In this section, we explain the notation we use in this paper and also mention two lemmas which we will refer to quite often through out this paper.

Lemma 2.1. ***One side Death Leap Principle*** *(One side DLP): if X is the position where the only possible move of Left is a jump and there is no two or more consecutive empty square in X then* $X \leq 0$*.*

Proof: We have to show that when Left moves first and two players take turn playing, Left will lose (Left will run out of the legal move first). This is true since after Left jumps over one of the F, Right can response by moving to the empty square where the F was jumped over.

Example 1)TTF□TTF□F ≤ 0.

Example 2) TTTF□F□TF ≤ 0.

Lemma 2.2. ***Blocking Rule*** *(BR): let X and Y be positions. The position* $XFFY \leq X + Y$*.*

Proof: Right has a choice not to move the two frogs. In this case, the value of the position $XFFY = X + Y$.

Convention:

We now explain our *shorthand* notation by using an example.

Example: To show: $\text{T}^a\text{F}\square\text{T}^k\text{FT}\square\text{F}^b \leq \frac{1}{2}$; $k \geq 0, a \geq 0, b \geq 1$.

We will have to show $\text{T}^a\text{F}\square\text{T}^k\text{FT}\square\text{F}^b - \frac{1}{2} \leq 0$.

That is to show $\text{T}^a\text{F}\square\text{T}^k\text{FT}\square\text{F}^b - \{0 \mid 1\} \leq 0$.

To show $G \leq 0$ we need to show that when Left moves first and two players take turn playing, Right will win. (On the other hand, to show $G \geq 0$ we show that when Right moves first and two players take turn playing, Left will win.)

We will show that in these two sum games, for all the possible choices of Left moves, Right can find a response so that he will win at the end.

We do some case analysis. In the above position Left has three choices. In the proof we will see

$$\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\mathrm{T}^k\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \overset{\overset{3}{\leftarrow}}{\leq} \tfrac{1}{2}\ .$$

These are the possible choices of Left. We write the response of Right immediately. You will see

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\square\mathrm{T}\mathrm{F}^b \leq 0$.
Explanation: Right responses by picking the left option of $\{0 \mid 1\}$ on the right hand side.

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\mathrm{F}\mathrm{T}\square\mathrm{F}^b \leq \frac{1}{2}$.
Explanation: Right responses by moving the left most F.

Case 3: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\mathrm{T}\mathrm{F}\square\mathrm{F}^{b-1} \leq 1$.
Explanation: Left picks the right option of $\{0 \mid 1\}$ on the right hand side. Right responses by moving the right most F.

Note:

1) When the position simplify to the known one in [3] or [5], we will claim such results without proving them again.

We state them here for clarity.

$$\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\square = a, \quad a \geq 0, b \geq 1. \tag{2.1}$$

$$\mathrm{T}^a\square\square\mathrm{F} = a-1, \quad a \geq 1. \tag{2.2}$$

$$\mathrm{T}^a\square\square\mathrm{T}\mathrm{F} = a, \quad a \geq 1. \tag{2.3}$$

2) When we consider the choices of move, we omit the move of an integer. By the number avoidance theorem (page 147 in [1]), number is an inferior choice than non-number. Amongst the choices of numbers, moving in an integer is the worst one.

3. $\mathrm{T}^a\square\square\square\mathrm{F}^b$

We show $\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

The values of the starting positions are of a great interest. Some of them have been first investigated in Erickson's paper [2]. The starting positions with the variables on both Toads and Frogs are even more interesting since they could not be proved by the symbolic finite-state method. In this section we show the value of $\mathrm{T}^a\square\square\square\mathrm{T}^b, a \geq 4, b \geq 4$. It is still an open problem about the values of the position $\mathrm{T}^a\square\square\square\square\mathrm{T}^b, a \geq 6, b \geq 6$ and $\mathrm{T}^a\square\square\square\square\square\mathrm{T}^b$.

The proof here is not long. However we hope to develop computer program to shorten and make the proof more routine in the future. The outline is shown below.

$$\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{FF}\square\mathrm{F}^{b-2} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FFF}\square\mathrm{F}^{b-3} = \{a-b \mid a-b\}.$$

Figure 1. $\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}$

Lemma 3.1. $\mathrm{T}^{a-3}\square\mathrm{T}^3\square\mathrm{F}^3\square\mathrm{F}^{b-3} = \quad \{a-b \mid a-b\}, \quad a \geq 4, b \geq 4.$

By symmetry, we only need to show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}} \square \overset{\xrightarrow{1}}{\mathrm{T}^3} \square\mathrm{F}^3\square\mathrm{F}^{b-3} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}.$

Case 1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}} \square\mathrm{TTF} \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{FF}\square\mathrm{F}^{b-3} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}.$

Case 1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TFF}\square\mathrm{F}^{b-3} \leq a-b.$

By BR, the left hand side $\leq \mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{T} + \square\mathrm{F}^{b-3} = a-3-(b-3) = a-b$. Refer to (2.1).

Case 1.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

By BR, the left hand side $\leq \mathrm{T}^{a-4}\square\mathrm{TTTFT}\square + \square\mathrm{F}^{b-3} = a-3-(b-3) = a-b$. Refer to (2.1).

Case 1.3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}}\ \square\mathrm{TTF}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

Case 1.3.1: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side $= \mathrm{T}^{a-3}\square + \square\mathrm{TF}\square\mathrm{F}^{b-3} = (a-3)-(b-3) = a-b$. Refer to (2.1).

Case 1.3.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFTFF}\square\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $(a-4)-2(b-3) = a-2b+2$.

Case 2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-4}}\ \square\mathrm{TTT}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 2.1: $\mathrm{T}^{a-4}\square\mathrm{TTTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.2.

Case 2.2: $\mathrm{T}^{a-5}\square\mathrm{TTTTTF}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $\leq (a-5)-(b-3) = a-b-2$. Refer to (2.1).

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTTF}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $\leq (a-3)-(b-3) = a-b$. Refer to (2.1).

Lemma 3.2. $\mathrm{T}^{a-2}\square\mathrm{T}^2\square\mathrm{F}^2\square\mathrm{F}^{b-2} = \ \{a-b \mid a-b\},\ \ a \geq 4, b \geq 4$.

By symmetry, we only need to show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\ \overset{\xrightarrow{1}}{\mathrm{T}^2}\ \square\mathrm{F}^2\square\mathrm{F}^{b-2} \leq\ \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\mathrm{TF}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F}^{b-2} \leq\ \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1.1: $\mathrm{T}^{a-2}\square\mathrm{TFFT}\square\square\mathrm{F}^{b-2} \leq\ \ \{a-b \mid a-b\}$.

Left has to move the left most T. Since otherwise right will move the left most F and block the left most position. So we have
$\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F}^{b-2} \leq a-b$.

The left hand side is $(a-3)-(b-3) = a-b$. Refer to (2.2).

Case 1.2: $\mathrm{T}^{a-3}\square\mathrm{TTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq \{a-b \mid a-b\}$.

This is case 1 in lemma 3.1.

Case 1.3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\,\square\mathrm{TF}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 1.3.1: $\mathrm{T}^{a-2}\square\mathrm{TFFT}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

Left has to move the left most T. Since otherwise right will move the left most F and block the left most position. So we have
$\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TTFFTF}\square\square\mathrm{F}^{b-3} \leq a-b$.
The left hand side is $(a-3)-(b-3) = a-b$. Refer to (2.3).

Case 1.3.2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\,\square\mathrm{TTF}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.
This is case 1.3 in lemma 3.1.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq \{a-b \mid a-b\}$, true by lemma 3.1.

Case 3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\,\square\mathrm{T}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 3.1: $\mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.3 above.

Case 3.2: $a-b \leq a-b$,   by lemma 3.1.

Lemma 3.3. $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

By symmetry, we only need to show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\,\square\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{a-b \overset{\overset{3}{\leftarrow}}{\mid} a-b\}$.

Case 1: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\,\square\square\mathrm{TFF}\square\mathrm{F}^{b-2} \leq \{a-b \overset{\overset{2}{\leftarrow}}{\mid} a-b\}$.

Case 1.1: $\mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{F}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$.
This case 1 of lemma 3.2.

Case 1.2: $\mathrm{T}^{a-1}\square\square\mathrm{TFFF}\square\mathrm{F}^{b-3} \leq a-b$.

$\Rightarrow \mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.
This is case 1.3 of lemma 3.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$, true by lemma 3.2.

Case 3: $\overset{\xrightarrow{2}}{\mathrm{T}}{}^{a-1}\,\square\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq a-b$.

Case 3.1: $\mathrm{T}^{a-1}\square\square\mathrm{TFFF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.2 above.

Case 3.2: $a-b \leq a-b$, by lemma 3.2.

Theorem 3.4. $\mathrm{T}^{a}\square\square\square\mathrm{F}^{b} = \{a-b \mid a-b\}$, $a \geq 4$, $b \geq 4$.

By symmetry, we only need to show $\overset{\xrightarrow{1}}{\mathrm{T}}{}^{a}\,\square\square\square\mathrm{F}^{b} \leq \{a-b \overset{\xleftarrow{2}}{\mid} a-b\}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{a-b \mid a-b\}$, true by lemma 3.3.

Case 2: $\mathrm{T}^{a}\square\square\mathrm{F}\square\mathrm{F}^{b-1} \leq a-b$.

$\Rightarrow a-b \leq a-b$, by lemma 3.3.

The theorem is proved. □

4. Future Work

Toads and Frogs is beautiful game with many nice patterns everywhere. The works done so far could have been just the tip of an iceberg. In making a progress, it is important to stress the role of computer program. Computer program will help seeing the bigger picture instead of getting lost in details. We have to wait to see whether anyone could find a better algorithm to automated or at least assisted the proof of the new results. The new algorithm might also as well be useful in other games or other brunches of mathematics.

Erickson made 5 conjectures in [1]. Four of them have already been confirmed, three are positive and one is negative. We prove two of them, conjecture 1 and 3, in this paper. Conjecture 2 has already been proved (in [3]) as a very special of the positions of the class □□□FF. Conjecture 4 is wrong (See [4]). Conjecture 5 is difficult and still open. The author made five new conjectures in [4]. Two of these new conjectures are the refine version of conjecture 5 of Erickson. We believe that proving these five new conjectures could lead to a new technique and a better understanding of this game.

We end this paper by stating conjecture and open problem similar to the positions we proved in this paper.

Conjecture:
$\mathrm{T}^a\square\square\square\square\mathrm{F}^b = \{a-b|a-b\}, \quad a \geq 6, b \geq 6.$

Open Problem:
For a fixed number $i \geq 5$, find the values of $\mathrm{T}^a\square^i\mathrm{F}^b$.

APPENDIX

Appendix A. $\mathrm{T}^a\square\square\mathrm{F}^b$

We show $\mathrm{T}^a\square\square\mathrm{F}^b = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \quad a > b \geq 2.$

This position is the smallest nontrivial starting position with variable on both Toads and Frogs. The values of this position are the first conjecture of Erickson in [2]. We will do the case analysis similar to the one in the previous section. The proof is not long but tricky.

For the case $a > b = 2$, the result is already in [3] and [5].
For the case $a > b \geq 3$,we will prove 11 lemmas that will lead to the main theorem.

We first show the value of $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1}$. Then we show the result by showing that $\mathrm{T}^a\square\square\mathrm{F}^b = \mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1}$. Below is how the tree looks like at the beginning.

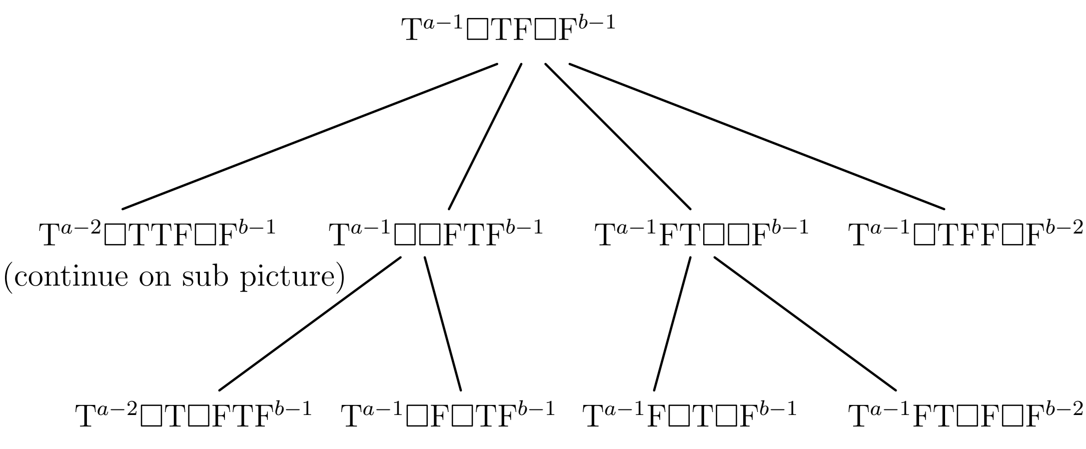

Figure 2. Main tree

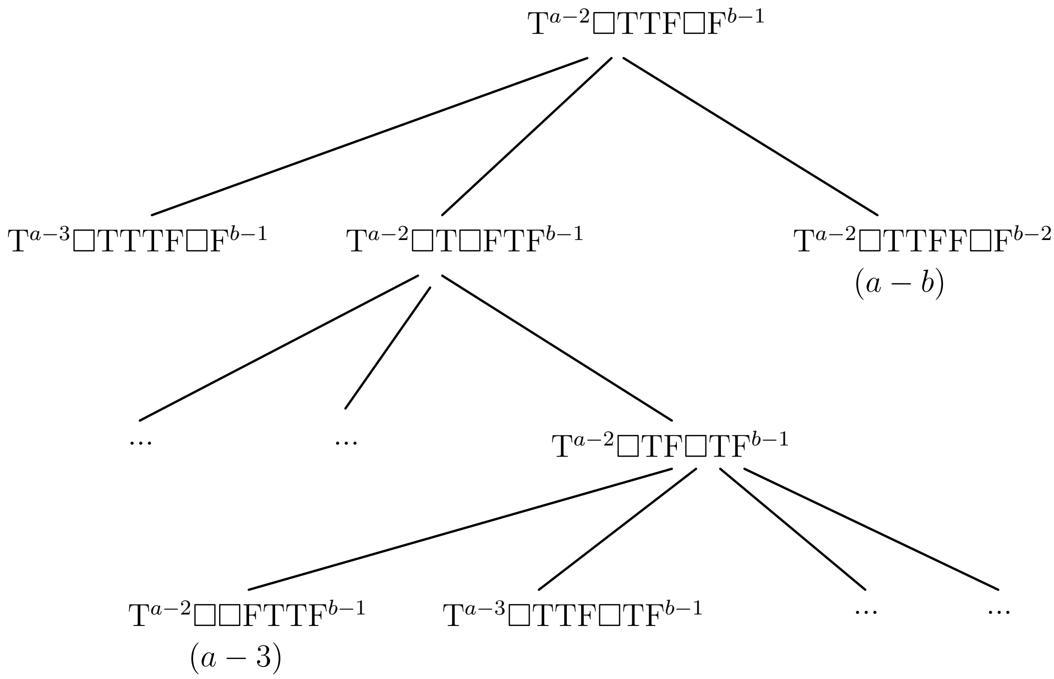

Figure 3. Sub picture of the main tree

Lemma A.1,A.2 and A.3 will be useful to prove the subsequence lemmas.

Lemma A.1. $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b = a, \quad k \geq 2, a \geq 0, b \geq 0.$

We show:

I) $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \leq a, \quad k \geq 2, a \geq 0, b \geq 0.$

II) $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \geq a, \quad k \geq 2, a \geq 0, b \geq 0.$

I) To show $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \leq a, \quad k \geq 2, a \geq 0, b \geq 0.$

By BR, the left hand side $\leq \mathrm{T}^a\square + \square\mathrm{TF}^b = a + 0 = a.$

II) To show $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \geq a, \quad k \geq 2, a \geq 0, b \geq 0.$

We show that Right moves first and Left always wins. Right has only one possible move.

$\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{FF}\square\mathrm{TF}^b \geq a$. The left hand side is $\mathrm{T}^a\square\ + \square\mathrm{TF}^b = a + 0 = a$.

Lemma A.2. $\mathrm{T}^a\square\mathrm{T}^k\square\mathrm{FTF}^b \geq a - 1, \quad k \geq 1, a \geq 1, b \geq 0.$

$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{T}^{k+1}\mathrm{F}\square\mathrm{TF}^b \geq a - 1.$
By BR, the left hand side $\geq \mathrm{T}^{a-1}\square + \square\mathrm{TF}^b = (a-1) + 0 = a - 1.$

Lemma A.3. $\stackrel{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\mathrm{T}^i\mathrm{F}^j\ \stackrel{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \leq \{0 \stackrel{\overset{3}{\leftarrow}}{|} 0\}, \quad i \geq 0, j \geq 1, a \geq 0, b \geq 1.$

Proof by induction on a: (I will omit the base case since it is the same as the induction step except there is no case 2).

Case1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^i\mathrm{F}^j\square\mathrm{TF}^b \leq 0$, true by one side DLP.

Case2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{i+1}\mathrm{F}^j\mathrm{T}\square\mathrm{F}^b \leq \{0 \mid 0\}$, true by induction.

Case3: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^i\mathrm{F}^j\mathrm{TF}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

Lemma A.4. $\mathrm{T}^a\mathrm{F}\ \stackrel{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^k\ \stackrel{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \leq 0, \quad k \geq 1, a \geq 0, b \geq 2.$

Case1: $\mathrm{T}^a\mathrm{F}\ \stackrel{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{k+1}\ \stackrel{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{b-1} \leq 0$

Case1.1: $\mathrm{T}^a\mathrm{FTF}\square\mathrm{F}^k\square\mathrm{TF}^{b-1} \leq 0$, true by one side DLP.

Case1.2: $\mathrm{T}^a\mathrm{F}\square\mathrm{TF}^{k+1}\mathrm{TF}\square\mathrm{F}^{b-2} \leq 0$, true by one side DLP.

Case2: $\mathrm{T}^a\mathrm{F}\square\mathrm{TF}^k\mathrm{TF}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

Lemma A.5. $\mathrm{T}^a\mathrm{F}\ \stackrel{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^k\square\mathrm{TF}^b \leq \{0 \stackrel{\overset{2}{\leftarrow}}{|} 0\}, \quad k \geq 0, a \geq 0, b \geq 3.$

Case1: $\mathrm{T}^a\mathrm{F}\square\mathrm{TF}^{k+1}\mathrm{T}\square\mathrm{F}^{b-1} \leq \{0 \mid 0\}$, true by lemma A.3.

Case2: $\mathrm{T}^a\mathrm{FT}\square\mathrm{F}^{k+1}\mathrm{T}\square\mathrm{F}^{b-1} \leq 0$, true by lemma A.4.

Lemma A.6 will be used for bypass reversible move in figure 3.

Lemma A.6. $\stackrel{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \square\ \stackrel{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF}^b \leq \mathrm{T}^a\square\mathrm{TTF}\square\ \stackrel{\overset{3}{\leftarrow}}{\mathrm{F}^b}, \quad a \geq b \geq 2.$

Case1: $\mathrm{T}^a\square\square\mathrm{F} \le \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$
$\Rightarrow a-1 \le \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$ since $\mathrm{T}^a\square\square\mathrm{F} = a-1$, by (2.2). True by lemma A.2.

Case2: $\mathrm{T}^{a-1}\square\mathrm{TTF}\square\mathrm{TF}^b \le \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$.
The left hand side is $a-1$ by lemma A.1.
Then the statement is true by lemma A.2.

Case3: $\mathrm{T}^a\mathrm{FT}\square\square\mathrm{TF}^b \le a-b+1$

$\Rightarrow \mathrm{T}^a\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{b-1} \le a-b+1$, true by lemma A.3.

Lemma A.7. $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\,\square\mathrm{TT}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F}^b \le a,\quad a \ge 0, b \ge 1.$

Case 1: $\mathrm{T}^a\square\mathrm{TTF}\square\mathrm{TF}^b \le a$, true by lemma A.1.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{TTTTFF}\square\mathrm{F}^{b-1} \le a$. The left hand side is $a-b$.

Lemma A.8 will be used for bypass reversible move in figure 5.

Lemma A.8. $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\,\square\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{TF}^b \le \mathrm{T}^a\square\square\,\overset{\overset{3}{\leftarrow}}{\mathrm{F}}\,\mathrm{TF}^b,\quad a \ge 2, b \ge 0.$

Case 1: $a-2 \le \mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{FTF}^b$, true by lemma A.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\mathrm{TF}^b \le \mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{FTF}^b$, the left hand side $= a-2$, by lemma A.1, true by lemma A.2.

Case 3: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{TF}^b \le \mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{TF}^b$. This is obviously true.

Lemma A.9. $\mathrm{T}^a\square\mathrm{F}\square\mathrm{TF}^b \le 1,\quad a \ge 2, b \ge 2.$

$\Rightarrow \mathrm{T}^{a-1}\mathrm{FT}\square\square\mathrm{TF}^b \le 1.$

$\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{b-1} \le 1$, true by lemma A.3.

Lemma A.10. $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\square\mathrm{F}^b = \{0 \mid 0\},\quad k \ge 1, a \ge 3, b \ge 2.$

Need to show:

I) $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\square\mathrm{F}^b \le \{0 \mid 0\},\quad k \ge 1, a \ge 0, b \ge 2.$

II) $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\square\mathrm{F}^b \geq \{0 \mid 0\}, \quad k \geq 1, a \geq 3, b \geq 1.$

I) To show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}^k}\ \square\mathrm{F}^b \leq\ \{0 \overset{\overset{3}{\leftarrow}}{\mid} 0\}, \quad k \geq 1, a \geq 0, b \geq 2.$

Prove by induction on a**:** (I will omit the base case since it is the same as the induction step except no case 2).

Induction step:

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^{k-1}\mathrm{F}\mathrm{T}\square\mathrm{F}^{b-1} \leq \{0 \mid 0\}$, true by lemma A.3.

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\square\mathrm{F}^b \leq \{0 \mid 0\}$, true by induction.

Case 3: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

II) To show $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}^b}\geq\{0 \overset{\overset{2}{\rightarrow}}{\mid} 0\}, \quad k \geq 1, a \geq 3, b \geq 1.$

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^{k-1}\square\mathrm{F}\mathrm{T}\mathrm{F}^{b-1} \geq \{0 \mid 0\}$, true by negative of lemma A.5.

Case 2: $\mathrm{T}^{a-1}\square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{T}^{k+1}\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}^b}\geq 0.$

Case 2.1: $\mathrm{T}^{a-1}\square\mathrm{F}\mathrm{T}^k\square\mathrm{F}\mathrm{T}\mathrm{F}^{b-1} \geq 0$, true by negative of lemma A.4.

Case 2.2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^k\square\mathrm{T}\mathrm{F}^b \geq 0$, true by one side DLP.

Lemma A.11. $\{* \mid \overset{\overset{1}{\rightarrow}}{1} - b\}\ \leq \mathrm{T}^a\square\mathrm{T}\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square\ \overset{\overset{3}{\leftarrow}}{\mathrm{F}^b}, \quad b \geq 1, a \geq b+2.$

Case 1: $* \leq a - b - 1.$

Case 2: $\{* \mid \overset{\overset{1}{\rightarrow}}{1} - b\}\leq \mathrm{T}^a\mathrm{F}\square\mathrm{T}\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \square\ \overset{\overset{3}{\leftarrow}}{\mathrm{F}^b}.$

Case 2.1: $* \leq \mathrm{T}^a\mathrm{F}\square\square\mathrm{F}\mathrm{T}\mathrm{F}^b$, true by the negative of lemma A.5.

Case 2.2: $1 - b \leq \mathrm{T}\square\square\mathrm{F}^b$. The right hand side is $1 - b$, by (2.2).

Case 2.3: $1 - b \leq \mathrm{T}^a\mathrm{F}\square\mathrm{T}\mathrm{F}\mathrm{F}\square\mathrm{F}^{b-1}$, true by the negative of lemma A.1.

Case 3: $\{* \mid 1-b\} \leq a-b$.

$$\Rightarrow 0 \leq a-b, \text{ which is true.}$$

After applying lemma 1,6,7,8,9,10,11 to the tree in figure 3, it looks like:

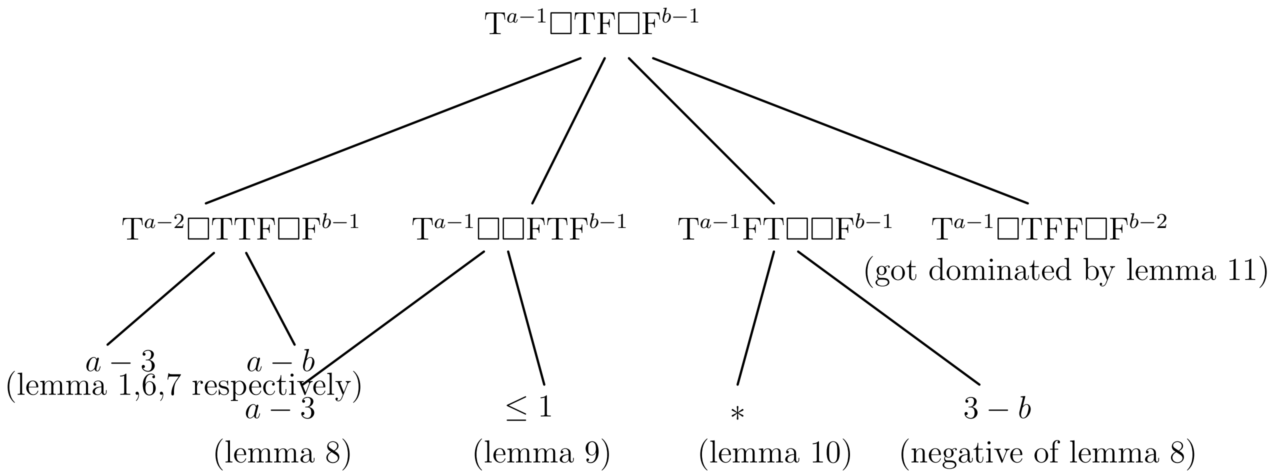

FIGURE 4. Tree after applying lemma 1,6,7,8,9,10,11

It might be helpful to provide some explanation of the value $a-3$ after applying lemma 8 above. We apply the bypass reversible move by comparing the position $T^{a-1}\square\square FTF^{b-1}$ and $T^{a-2}\square TF\square TF^{b-1}$ which is the only right option of $T^{a-2}\square T\square FTF^{b-1}$. The left options of $T^{a-2}\square TF\square TF^{b-1}$ are $T^{a-2}\square\square FTTF^{b-1}$ and $T^{a-3}\square TTF\square TF^{b-1}$ in which both have values $a-3$.

After applying these lemmas, we have
$T^{a-1}\square TF\square F^{b-1} = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$.

We finally show the main theorem using the result above.

Theorem A.12. $T^{a}\square\square F^{b} = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\},\ \ a > b \geq 3.$

Need to show:

I) $T^{a}\square\square F^{b} \leq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\},\ \ a > b \geq 3.$

II) $\mathrm{T}^a\square\square\mathrm{F}^b \geq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \ \ a > b \geq 3.$

I) To show $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^a}\square\square\mathrm{F}^b \leq \{\{a-3 \mid a-\overset{\overset{2}{\leftarrow}}{b}\} \mid \{* \mid 3-b\}\}, \ \ a > b \geq 3.$

Case 1: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} \leq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, true by the tree above.

Case 2: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^a}\square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{* \mid 3 \overset{\overset{2}{\leftarrow}}{-} b\}.$

Case 2.1: $\{* \mid 3-b\} \leq \{* \mid 3-b\}$, true by the tree above.

Case 2.2: $\mathrm{T}^a\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq 3-b.$
$\Rightarrow \mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{FTF}^{b-1} \leq 3-b$, true by lemma A.2.

II) To show $\mathrm{T}^a\square\square\overset{\overset{1}{\leftarrow}}{\mathrm{F}^b} \ \geq \ \{\{a-3 \mid a-\overset{\overset{2}{\rightarrow}}{b}\} \mid \{* \mid 3-b\}\}, \ \ a > b \geq 3.$

Case 1: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} \geq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, true by the tree above.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{T}\square\overset{\overset{1}{\leftarrow}}{\mathrm{F}^b} \ \geq \ \{a-3 \overset{\overset{2}{\rightarrow}}{\mid} a-b\}.$

Case 2.1: $\{a-3 \mid a-b\} \geq \{a-3 \mid a-b\}$, true by the tree above.

Case 2.2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{F}^b \geq a-3$. This is the negative of case **I)** 2.2.

The theorem is proved. □

Appendix B. $\mathrm{T}^a\square\square\square\mathrm{FFF}$

We show $\mathrm{T}^a\square\square\square\mathrm{FFF} = a - \frac{7}{2}$, $a \geq 5$.

We are supposed to be able to prove the values of the class □□□FFF by the symbolic finite-state method (see [3],[5]). Then the result above will follow as one of the special case of many positions in the class. However there are too many possible cases to make conjectures. For now we prove the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}$, $a \geq 5$ by hand. The proof is quite lengthy but the plan is clear. We need 7 lemmas before we can show the main theorem.

Below is the outline.

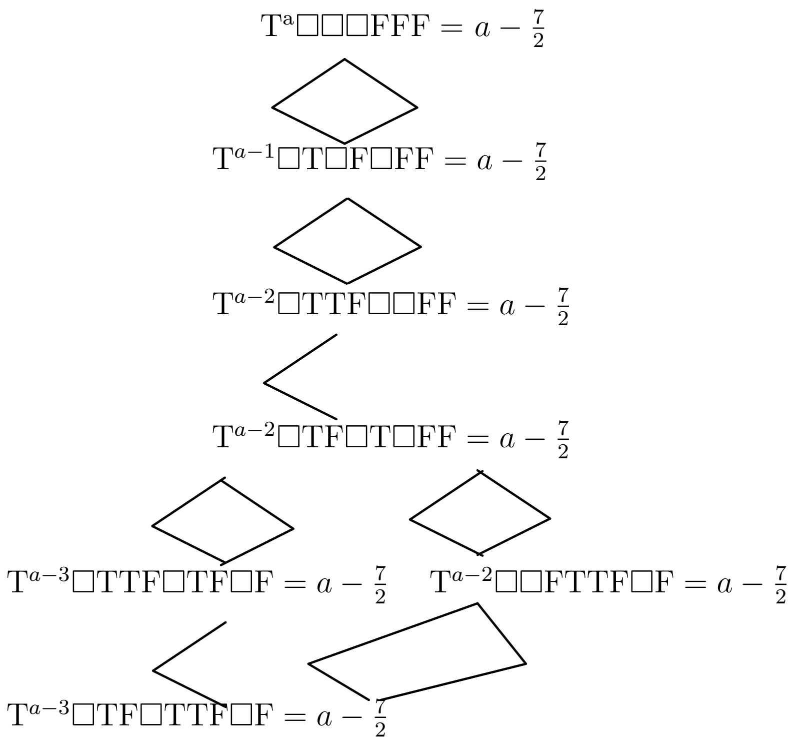

Figure 5. Main tree

We first show lemma B.1. Then we will start to work from the bottom of the main tree (figure 5) and goes all the way up to the top.

Lemma B.1. $T^{a-3}\square TF\square TF\square TF \leq a - \frac{7}{2}$, $a \geq 5$.

To Show $\overset{\overset{3}{\rightarrow}}{T^{a-3}} \square \overset{\overset{2}{\rightarrow}}{T} F\square \overset{\overset{1}{\rightarrow}}{T} F\square TF \leq \overset{\overset{4}{\leftarrow}}{a - \frac{7}{2}}$.

Case 1: $T^{a-3}\square TF\square\square FTTF \leq a - 4$.

The left hand side is $T^{a-3}\square TF\square\square F = a - 4, \;\; a \geq 4$. We know this value from the symbolic finite-state method we developed in [3] (Also see [5]).

Case 2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}}$ □□FT $\overset{\xrightarrow{1}}{\mathrm{T}}$ F□TF $\leq a-4$.

Case 2.1: T^{a-3}□□FTF□ $\leq a-4$. The left hand side already is $a-4$.

Case 2.2: T^{a-4}□TF□TTF□TF $\leq a-4$, true by one side DLP.

Case3: $\overset{\xrightarrow{3}}{\mathrm{T}^{a-4}}$ □T $\overset{\xrightarrow{2}}{\mathrm{T}}$ F□ $\overset{\xrightarrow{1}}{\mathrm{T}}$ F□TF $\leq a-4$.

Case 3.1: T^{a-4}□TTF□F□ $\leq a-4$. The left hand side is $\{\{a-4 \mid a-4\} \mid a-4\}$.

Case 3.2: T^{a-4}□TF□TTF□TF $\leq a-4$, true by one side DLP.

Case 3.3: T^{a-5}□TTTFFT□□TF $\leq a-4$. The left hand side is $(a-5)+1$.

Case 4: T^{a-3}F $\overset{\xrightarrow{2}}{\mathrm{T}}$ □□ $\overset{\xrightarrow{1}}{\mathrm{T}}$ F□TF $\leq a-3$.

Case 4.1: T^{a-3}FT□□F□ $\leq a-3$. The left hand side is $\{\{a-3 \mid a-3\} \mid a-3\}$.

Case 4.2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}}$ F□TF $\overset{\xrightarrow{1}}{\mathrm{T}}$ □□TF $\leq a-3$.

Case 4.2.1: T^{a-3}FFT□□T□TF $\leq a-3$.
The left hand side is $\{3 \mid 3\}$ which assure the statement for $a \geq 7$.

Case 4.2.2: T^{a-4}F□TTFT□□TF $\leq a-3$.
$\Rightarrow \mathrm{T}^{a-2}$FT□□TF $\leq a-3$, The left hand side is 1.

We now have lemma B.1. We are one step closer to the main theorem. We now prove the statement at the bottom of the tree.

Lemma B.2. T^{a-3}□TF□TTF□F $= a-\frac{7}{2}$, $a \geq 5$.

Need to show:

I) $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{TF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{3}{\rightarrow}}{\mathrm{T}^{a-3}} \square \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{T}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F} \leq \overset{\overset{4}{\leftarrow}}{a - \frac{7}{2}}$.

Case1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a - \frac{7}{2}$, true by lemma B.1.

Case 2: $\mathrm{T}^{a-3}\square\square\mathrm{FTTTFF}\square \leq a - \frac{7}{2}$. The left hand side is $a - 4$.

Case 3: $\mathrm{T}^{a-4}\square\mathrm{TTF}\square \leq a - \frac{7}{2}$. The left hand side is $a - 4$.

Case 4: $\mathrm{T}^{a-3}\square\mathrm{TF}\square \leq a - 3$. The left hand side is $a - 3$.

II) To Show $\mathrm{T}^{a-3}\square\mathrm{T}\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\square\mathrm{TTF}\square\,\overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \overset{\overset{3}{\rightarrow}}{a - \frac{7}{2}}$.

Case1: $\mathrm{T}^{a-3}\mathrm{F}\square\mathrm{T}\square\mathrm{TTF}\square\mathrm{F} \geq a - \frac{7}{2}$. The left hand side is $\geq 1 + (a - 3) - 1 = a - 3$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square \geq a - 3$. The left hand side is $a - 3$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\square\mathrm{T}\square\,\overset{\overset{2}{\leftarrow}}{\mathrm{F}}\,\mathrm{TF} \geq a - 4$.

Case 3.1: $\mathrm{T}^{a-3}\mathrm{F}\square\mathrm{T}\square\mathrm{T}\square\mathrm{FTF} \geq a - 4$.
$\Rightarrow \mathrm{T}^{a-3}\mathrm{F}\square\square\mathrm{TTF}\square\mathrm{TF} \geq a-4$. The left hand side is $\geq a-3$, by BR.

Case 3.2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\square\mathrm{FTTF} \geq a - 4$.
The left hand side is $a - 4$.

We prove lemma B.2.

Lemma B.3. $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} = a - \frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\ \square\square\mathrm{FT}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F} \leq\ \overset{\overset{3}{\leftarrow}}{a} - \frac{7}{2}$.

Case1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\ \square\square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq \overset{\overset{3}{\leftarrow}}{a} - \frac{7}{2}$.

Case 1.1: $\mathrm{T}^{a-2}\square\square\mathrm{F}\square\mathrm{F} \leq a-4$. The left hand side is $a-4$.

Case 1.2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a-\frac{7}{2}$, true by lemma B.1.

Case 1.3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\ \square\mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq a-3$.

Case 1.3.1: $\mathrm{T}^{a-2}\mathrm{F}\square\square\square\mathrm{F} \leq a-3$. The left hand side is $a-3$.

Case 1.3.2: $\mathrm{T}^{a-3}\mathrm{FT}\square\square\mathrm{TF}\square\mathrm{TF} \leq a-3$. This is case 4 of lemma B.1.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by lemma B.2.

Case 3: $\mathrm{T}^{a-2}\square\square\mathrm{FTTFF}\square \leq a-3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-2}\square\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TTF}\square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq\ \overset{\overset{3}{\rightarrow}}{a} - \frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \geq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by lemma B.2.

Case 2: $\mathrm{T}^{a-2}\square\square\mathrm{F} \geq a-3$. The left hand side is $a-3$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TTF}\square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \geq a-4$.

Case 3.1: $a-4 \geq a-4$, by lemma B.2.

Case 3.2: $\mathrm{T}^{a-3}\square\square\mathrm{TF} \geq a-4$. The left hand side is $a-3$.

We have lemma B.3 here. Lemma B.4 is similar to lemma B.3. They are also at the same level in the picture.

Lemma B.4. $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} = a-\frac{7}{2}$, $a \geq 5$.

Need to show :

I) $T^{a-3}\square TTF\square TF\square F \le a - \frac{7}{2}$.

II) $T^{a-3}\square TTF\square TF\square F \ge a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{3}}{T^{a-3}}\,\square T\,\overset{\xrightarrow{2}}{T}\,F\square\,\overset{\xrightarrow{1}}{T}\,F\square F \le \overset{\xleftarrow{4}}{a -} \frac{7}{2}$.

Case 1: $\overset{\xrightarrow{2}}{T^{a-3}}\,\square T\,\overset{\xrightarrow{1}}{T}\,F\square F\square TF \le \overset{\xleftarrow{3}}{a -} \frac{7}{2}$.

Case 1.1: $T^{a-3}\square TF\square TF\square TF \le a - \frac{7}{2}$, true by lemma B.1.

Case 1.2: $T^{a-4}\square TTTFF\square\square TF \le a - \frac{7}{2}$. The left hand side is $(a-4)-1$.

Case 1.3: $T^{a-3}\square TTFF\square\square TF \le a-3$. The left hand side is $(a-3)-1$.

Case 2: $T^{a-3}\square TF\square TTF\square F \le a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by lemma B.2.

Case 3: $T^{a-4}\square TTTFFT\square\square F \le a - \frac{7}{2}$. The left hand side is $a-4$.

Case 4: $T^{a-3}\square TTFFT\square\square F \le a-3$. The left hand side is $a-3$.

II) To Show $T^{a-3}\square TTF\square T\,\overset{\xleftarrow{1}}{F}\,\square\,\overset{\xleftarrow{2}}{F} \ge \overset{\xrightarrow{3}}{a -} \frac{7}{2}$.

Case 1: $T^{a-3}\square TTFFT\square\square F \ge a-3$. The left hand side is $a-3$.

Case 2: $T^{a-3}\square T\square FTTFF\square \ge a-\frac{7}{2}$. The left hand side is $\{a-3 \mid a-3\}$.

Case 3: $T^{a-3}\square T\square FTTF\square F \ge a-4$. This is II) case 3 of lemma B.3.

Lemma B.5. $T^{a-2}\square TF\square T\square FF = a - \frac{7}{2}$, $a \ge 5$.

Need to show :

I) $T^{a-2}\square TF\square T\square FF \le a - \frac{7}{2}$.

II)) $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} \geq a-\frac{7}{2}$.

I) To Show $\overset{\xrightarrow{3}}{\mathrm{T}^{a-2}} \square \overset{\xrightarrow{2}}{\mathrm{T}}\,\mathrm{F}\square \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{FF} \leq \overset{\xleftarrow{4}}{a-\frac{7}{2}}$.

Case1: $\mathrm{T}^{a-2}\mathrm{F} \overset{\xrightarrow{1}}{\mathrm{T}} \square\square\square\mathrm{TFF} \leq \overset{\xleftarrow{2}}{a-\frac{7}{2}}$.

Case 1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{FT}\square\mathrm{F} \leq a-\frac{7}{2}$.
The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\square\mathrm{FT}\square\mathrm{F} = \{1 \mid 1\}$.

Case 1.2: $\mathrm{T}^{a-2}\mathrm{F} \overset{\xrightarrow{2}}{\mathrm{T}} \square\square\mathrm{F} \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F} \leq a-3$.

Case 1.2.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{F}\square\square\mathrm{TF} \leq a-3$.
The left most position will get block eventually.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{F} \leq a-3$.

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} \leq a-3$.
The left hand side is $\{\{\{a-3 \mid 2\} \mid 1\} \mid\mid 0\}$.

Case 2: $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by lemma B.3.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by lemma B.4.

Case 4: $\overset{\xrightarrow{3}}{\mathrm{T}^{a-2}} \square \overset{\xrightarrow{2}}{\mathrm{T}}\,\mathrm{F}\square \overset{\xrightarrow{1}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F} \leq a-3$.

Case 4.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\square\square\mathrm{FTF} \leq a-3$.

$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{F}\square\mathrm{TF} \leq a-3$.

$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{F}\square\mathrm{TF} \leq a-3$., the left hand side is 1.

Case 4.2: $\mathrm{T}^{a-2}\square\square\mathrm{F} \leq a-3$. , the left hand side is $a-3$.

Case 4.3: $a-3 \leq a-3$, true by lemma B.4.

II) To Show $\mathrm{T}^{a-2}\Box\mathrm{T}\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\Box\mathrm{T}\Box\overset{\overset{2}{\leftarrow}}{\mathrm{F}}\mathrm{F} \geq \overset{\overset{3}{\rightarrow}}{a-\tfrac{7}{2}}$.

Case 1: $\mathrm{T}^{a-2}\mathrm{F}\Box\mathrm{T}\Box\mathrm{T}\Box\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\mathrm{F} \geq \overset{\overset{2}{\rightarrow}}{a-\tfrac{7}{2}}$.

Case 1.1: $\mathrm{T}^{a-2}\mathrm{F}\Box\Box\mathrm{TTF}\Box\mathrm{F} \geq a-\frac{7}{2}$. The left hand side is $\geq (a-2)-1$.

Case 1.2: $\mathrm{T}^{a-2}\Box\Box\mathrm{TT}\Box\mathrm{FF} \geq a-4$. The left hand side is $\geq (a-2)-2$.

Case 2: $\mathrm{T}^{a-2}\mathrm{F}\Box\Box\mathrm{FTTF}\Box\mathrm{F} \geq a-\frac{7}{2}$, true by lemma B.3.

Case 3: $\mathrm{T}^{a-3}\Box\mathrm{TTF}\Box\mathrm{T}\Box\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\mathrm{F} \geq a-4$.

$\Rightarrow a-4 \geq a-4$, by lemma B.4.

Lemma B.6. $\mathrm{T}^{a-2}\Box\mathrm{TTF}\Box\Box\mathrm{FF} = a-\frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a-2}\Box\mathrm{TTF}\Box\Box\mathrm{FF} \leq a-\frac{7}{2}$.

II) $\mathrm{T}^{a-2}\Box\mathrm{TTF}\Box\Box\mathrm{FF} \geq a-\frac{7}{2}$.

I) To Show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\Box\mathrm{T}\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\Box\Box\mathrm{FF} \leq \overset{\overset{3}{\leftarrow}}{a-\tfrac{7}{2}}$.

Case 1: $\mathrm{T}^{a-2}\Box\mathrm{TF}\Box\mathrm{T}\Box\mathrm{FF} \leq a-\frac{7}{2}$, true by lemma B.5.

Case 2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\Box\mathrm{TT}\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\Box\mathrm{F}\Box\mathrm{F} \leq \overset{\overset{3}{\leftarrow}}{a-\tfrac{7}{2}}$.

Case 2.1: $\mathrm{T}^{a-3}\Box\mathrm{TTF}\Box\mathrm{TF}\Box\mathrm{F} \leq a-\frac{7}{2}$, true by lemma B.4.

Case 2.2: $\mathrm{T}^{a-4}\Box\mathrm{TTTTFF}\Box\Box\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $(a-4)-2$.

Case 2.3: $\mathrm{T}^{a-3}\Box\mathrm{TTTFF}\Box\Box\mathrm{F} \leq a-3$. The left hand side is $(a-3)-2$.

Case 3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\,\Box\mathrm{T}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\mathrm{F}\Box\mathrm{F}\Box\mathrm{F} \le a-3$.

Case 3.1: $\mathrm{T}^{a-2}\Box\mathrm{TF}\Box\mathrm{TF}\Box\mathrm{F} \le a-3$. This is case I)4 of lemma B.5.

Case 3.2: $\mathrm{T}^{a-3}\Box\mathrm{TTTFF}\Box\Box\mathrm{F} \le a-3$. The left hand side is $(a-3)-2$.

II) To Show $\mathrm{T}^{a-2}\Box\mathrm{TTF}\Box\Box\,\overset{\xleftarrow{1}}{\mathrm{F}}\,\mathrm{F} \ge \overset{\xrightarrow{2}}{a-\tfrac{7}{2}}$.

Case 1: $\mathrm{T}^{a-2}\Box\mathrm{T}\Box\,\overset{\xleftarrow{1}}{\mathrm{F}}\,\mathrm{TF}\Box\,\overset{\xleftarrow{2}}{\mathrm{F}} \ge \overset{\xrightarrow{3}}{a-\tfrac{7}{2}}$.

Case 1.1: $\mathrm{T}^{a-3}\Box\mathrm{TTF}\Box\mathrm{TF}\Box\mathrm{F} \ge a-\frac{7}{2}$, true by lemma B.4.

Case 1.2: $\mathrm{T}^{a-2}\Box\Box\mathrm{TFTFF}\Box \ge a-\frac{7}{2}$.
The left hand side is $\ge \mathrm{T}^{a-2}\Box\mathrm{F}\Box = a-\frac{5}{2}$.

Case 1.3: $\mathrm{T}^{a-2}\Box\Box\mathrm{TFTF}\Box\mathrm{F} \ge a-4$.
The left hand side is $\ge \mathrm{T}^{a-2}\Box\mathrm{F}\Box - 1 = (a-\frac{5}{2})-1$.

Case 2: $\mathrm{T}^{a-2}\Box\mathrm{T}\Box\,\overset{\xleftarrow{1}}{\mathrm{F}}\,\mathrm{T}\Box\,\overset{\xleftarrow{2}}{\mathrm{F}}\,\mathrm{F} \ge a-4$.

Case 2.1: $a-4 \ge a-4$, by lemma B.5.

Case 2.2: $\mathrm{T}^{a-2}\Box\Box\mathrm{TFTF}\Box\mathrm{F} \ge a-4$. This is case 1.3 above.

Lemma B.7. $\mathrm{T}^{a-1}\Box\mathrm{T}\Box\mathrm{F}\Box\mathrm{FF} = a-\frac{7}{2}$, $a \ge 5$.

Need to show :

I) $\mathrm{T}^{a-1}\Box\mathrm{T}\Box\mathrm{F}\Box\mathrm{FF} \le a-\frac{7}{2}$.

II) $\mathrm{T}^{a-1}\Box\mathrm{T}\Box\mathrm{F}\Box\mathrm{FF} \ge a-\frac{7}{2}$.

I) To Show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}\,\Box\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\Box\mathrm{F}\Box\mathrm{FF} \le \overset{\xleftarrow{3}}{a-\tfrac{7}{2}}$.

Case 1: $\overset{\xrightarrow{1}}{\mathrm{T}^{a-1}}\,\Box\Box\mathrm{TFF}\Box\mathrm{F} \le \overset{\xleftarrow{2}}{a-\tfrac{7}{2}}$.

Case 1.1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\,\square\mathrm{TF}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\mathrm{F}\square\mathrm{F} \leq \overset{\xleftarrow{3}}{a - \frac{7}{2}}$.

Case 1.1.1: $\mathrm{T}^{a-2}\mathrm{F}\,\overset{\xrightarrow{2}}{\mathrm{T}}\,\square\square\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F} \leq \overset{\xleftarrow{3}}{a - \frac{7}{2}}$. (The left hand side is $\{1 \mid \frac{1}{2}\}$)

Case 1.1.1.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\square\square\mathrm{FTF} \leq a - 4$.

$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{F}\square\mathrm{TF} \leq a - 4$.

$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{F}\square\mathrm{TF} \leq a - 4$. The left hand side is 1.

Case 1.1.1.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFT}\square\square\mathrm{F} \leq a - \frac{7}{2}$.

$\Rightarrow \mathrm{T}^{a-1}\mathrm{FT}\square\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is 0.

Case 1.1.1.3: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{FT}\square\square\mathrm{F} \leq a - 3$. The left hand side is ≤ 2.

Case 1.1.2: $\mathrm{T}^{a-3}\square\mathrm{TTFTF}\square\square\mathrm{F} \leq a - \frac{7}{2}$.

The left hand side is $\leq (a-3) + \square\mathrm{T}\square\mathrm{F} = (a-3) - \frac{1}{2}$

Case 1.1.3: $\mathrm{T}^{a-2}\mathrm{F}\,\overset{\xrightarrow{2}}{\mathrm{T}}\,\square\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\mathrm{F}\square\mathrm{F} \leq a - 3$.

Case 1.1.3.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{FT}\square\square\mathrm{F} \leq a - 3$. This is case 1.1.1.3 above.

Case 1.1.3.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TTF}\square\square\mathrm{F} \leq a - 3$.

$\Rightarrow$ The left hand side is $\leq \mathrm{T}^{a}\mathrm{F}\square\square\mathrm{F} = \{\{1 \mid 1\} \mid\mid 0\}$.

Case 1.2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}\,\square\mathrm{F}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\mathrm{F}\square\mathrm{F} \leq a - 3$.

Case 1.2.1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}\,\square\mathrm{FF}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\square\mathrm{F} \leq a - 3$.

Case 1.2.1.1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}$ F□F□ $\overset{\xrightarrow{1}}{\mathrm{T}}$ □F $\leq a-3$.

Case 1.2.1.1.1: T^{a-1}FF□□□TF $\leq a-3$. The left hand side is -2.

Case 1.2.1.1.2: The left hand side goes to $\Rightarrow$ T^{a-1}F□T□F $\leq a-3$.

The left hand side is $\{\{\{a-3 \mid 2\} \mid 1\} \mid\mid 0\}$.

Case 1.2.1.2: T^{a-2}F $\overset{\xrightarrow{2}}{\mathrm{T}}$ □F $\overset{\xrightarrow{1}}{\mathrm{T}}$ □□F $\leq a-3$.

Case 1.2.1.2.1: T^{a-2}FTF□□T□F $\leq a-3$.

$\Rightarrow$ □T□T□F $= \{1 \mid 1\}$.

Case 1.2.1.2.2: T□T□□F $\leq a-3$. The left hand side is 2.

Case 1.2.2: T^{a-2}FT□T□F□F $\leq a-3$. This is case 1.1.3 above.

Case 2: T^{a-2}□TTF□□FF $\leq a-\frac{7}{2}$, true by lemma B.6.

Case 3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}$ □ $\overset{\xrightarrow{1}}{\mathrm{T}}$ F□□FF $\leq a-3$.

Case 3.1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}$ □F□ $\overset{\xrightarrow{1}}{\mathrm{T}}$ □FF $\leq a-3$.

Case 3.1.1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}$ □F□F $\overset{\xrightarrow{1}}{\mathrm{T}}$ □F $\leq a-3$.

Case 3.1.1.1: T^{a-1}F□□F□TF $\leq a-3$.

$\Rightarrow$ T^{a-1}□F□TF $= 1$.

Case 3.1.1.2: T^{a-2}F $\overset{\xrightarrow{2}}{\mathrm{T}}$ □□F $\overset{\xrightarrow{1}}{\mathrm{T}}$ □F $\leq a-3$.

Case 3.1.1.2.1: T^{a-2}FT□F□□TF $\leq a-3$.
The statement above is true since the left most part of the position gets block eventually.

Case 3.1.1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{F} \leq a-3$.

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} = \{\{a-3 \mid 2\} \mid 1\} \mid\mid 0\}$

Case 3.1.2: $a-3 \leq a-3$, by lemma B.5.

Case 3.2: $a-3 \leq a-3$, by lemma B.6.

II) To Show $\mathrm{T}^{a-1}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \mathrm{F} \geq a \overset{\overset{3}{\rightarrow}}{-} \frac{7}{2}$.

Case 1: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} \geq a-\frac{7}{2}$, true by lemma B.6.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \mathrm{F}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq a \overset{\overset{3}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.1: $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{FTF}\square\mathrm{F} \geq a-\frac{7}{2}$. This is II) case1 of lemma B.6.

Case 2.2: $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{T} \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \mathrm{FF}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1: $\mathrm{T}^{a-3}\square\mathrm{TTFT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \mathrm{F}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \mathrm{TF}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1.1.1: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square \geq a-3$.
The left hand side is $(a-3)+0 = a-3$.

Case 2.2.1.1.2: $\mathrm{T}^{a-3}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \mathrm{TFTF}\square \geq a-4$.
$\Rightarrow \mathrm{T}^{a-3}\square\square\mathrm{FTTFTF}\square = a-4$.
The left hand side $\geq \mathrm{T}^{a-3}\square\square\mathrm{F}+\mathrm{FTF}\square = (a-4)+0 = a-4$.

Case 2.2.1.2: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TFF}\square \geq a-4$.
The left hand side $\geq \mathrm{T}^{a-3}\square+\square\mathrm{TFF}\square = (a-3)+0 = a-3$.

Case 2.2.2: $\mathrm{T}^{a-2}\square\square\mathrm{TTFFF}\square \geq a-4$. The left hand side is $2(a-2)$.

Case 2.3: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \ a-4.$

Case 2.3.1: $\mathrm{T}^{a-3}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TF}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \ a-4.$

Case 2.3.1.1: $a-4 \geq a-4$, by case II 1.1 of lemma B.6.

Case 2.3.1.2: $\mathrm{T}^{a-3}\square\mathrm{T}\square\mathrm{TFTFF}\square \geq a-4.$

The left hand side $\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TF}\square = a-3.$

Case 2.3.2: $\mathrm{T}^{a-3}\square\mathrm{TT}\square\mathrm{TFFF}\square \geq a-4.$
The left hand side $\geq a-3.$

Case 3: $\mathrm{T}^{a-2}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \square \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F} \geq \ a-4.$

Case 3.1: $a-4 \geq a-4$, by lemma B.6.

Case 3.2: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FF}\square\mathrm{F} \geq a-4$, true by the case 2.3

Theorem B.8. $\mathrm{T}^{a}\square\square\square\mathrm{FFF} = a-\frac{7}{2}$, $a \geq 5$.

Need to show :

I) $\mathrm{T}^{a}\square\square\square\mathrm{FFF} \leq a-\frac{7}{2}.$

II) $\mathrm{T}^{a}\square\square\square\mathrm{FFF} \geq a-\frac{7}{2}.$

I) To Show $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}}\ \square\square\square\mathrm{FFF} \leq \ \overset{\overset{2}{\leftarrow}}{a-\frac{7}{2}}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \leq a-\frac{7}{2}$, true by lemma B.7.

Case 2: $\mathrm{T}^{a}\square\square\mathrm{F}\square\mathrm{FF} \leq a-3.$

$\Rightarrow a-3 \leq a-3$, by lemma B.7.

II) To Show $\mathrm{T}^{a}\square\square\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{FF} \geq \ \overset{\overset{2}{\rightarrow}}{a-\frac{7}{2}}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FFF} \geq a-\frac{7}{2}$, true by lemma B.7.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{T}\square\square\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\mathrm{FF} \geq \quad a-4.$

$\Rightarrow a-4 \geq a-4$, by lemma B.7.

The main theorem is proved. □

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