FURTHER HOPPING WITH TOADS AND FROGS

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Abstract. We show the new results of the combinatorial game "Toads and Frogs". We present new values of starting positions. We also discuss the values of all positions with exactly one □ and the positions T^a□□F^a. At the end, we post five new conjectures and discuss the possible future work.

1. Introduction

The game *Toads and Frogs*, invented by Richard Guy, is extensively discussed in "Winning Ways"[1], the famous classic by Elwyn Berlekemp, John Conway, and Richard Guy, that is the *bible* of combinatorial game theory. This game got so much coverage because of the simplicity and elegance of its rules, the beauty of its analysis, and as an example of a combinatorial game whose positions do not always have values that are numbers.

The game is played on a $1 \times n$ strip with either Toad(T) , Frog(F) or □ on the squares. Left plays T and Right plays F. T may move to the immediate square on its right, if it happens to be empty, and F moves to the next empty square on the left, if it is empty. If T and F are next to each other, they have an option to jump over one another, in their designated directions, provided they land on an empty square [1, p.14].

Already in [1] there is some analysis of Toads and Frogs positions, but on *specific*, small boards, such as TTT□FF. In 1996, Erickson [2] analyzed more general positions, and made six conjectures about the values of some families of positions. All of them are starting positions (positions where all T are rightmost and all F are leftmost). Erickson's conjectures were:

E1: T^a□□F^b $= \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$ for all $a > b \geq 2$.
E2: T^a□□□FF $= \{a-2 \mid a-2\}$ for all $a \geq 2$.
E3: T^a□□□FFF $= a - \frac{7}{2}$ for all $a \geq 5$.
E4: T^a□^{a}F^{a-1} $= 1$ or $\{1 \mid 1\}$ for all $a \geq 1$.
E5: T^a□^{b}F^a is an infinitesimal for all a, b except (a, b) =(3,2).
E6: Toads and Frogs is NP-hard.

Jesse Hull proved E6 in 2000 [3]. Doron Zeilberger and the author proved E2 in [4]. The proofs of Conjectures E1 and E3 are in [5]. The results of this paper include a counterexample to E4. Conjecture E5 is still open.

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This paper is a sequel to [4], which discussed the symbolic finite-state approach to prove the value of the positions in class A and B (defined below). However, the patterns of the value of Toads and Frogs game are not limited to only class A or class B. There are also patterns in the positions where variables are on both T and F, for example $\mathrm{T}^a\square\square\mathrm{F}^b$. In this paper we analyze some of these positions.

Definitions:

Class A: All positions that have a *fixed* number of occurrences of □ and F, but a *variable* (symbolic) number of T's in between the □'s and F's.

Class B: All positions that have a *fixed* number of occurrences of T's and F's, but a *variable* (symbolic) number of □'s in between the T's and F's.

General Class Ai: all positions with a *fixed* number, (numeric) i, occurrences of □ (with symbols on both T and F).

General Class Bi: all positions with a *fixed* number, (numeric) i, occurrences of F (with symbols on both T and □).

For the general classes Ai and Bi, we can not apply the finite state method used in [4] anymore, since we now have infinitely many positions that come from the combination of the two letters with symbols on them. However we are able to categorize all positions in Class A1, the class of all positions with exactly one □. It is in fact the only general class that we manage to do it for.

Many positions in these general classes do not have a nice compact formula; for example in A2, $\mathrm{T}^a\square\mathrm{TF}\square\mathrm{TF}^b$. On the other hand, many positions have a nice formula. Once we detect the patterns of the positions, the proof is quite routine. We then do the proof of each specific position by hand with the help of a computer. We hope to see the computer playing more roles in assisting with proofs in the future.

The main results in this paper are:

Theorem 1. *The values of positions in class A1, any position with one □, can be calculated recursively from 8 lemmas.*

Theorem 2. $\mathrm{T}^a\square\square\mathrm{F}^a$ *is an infinitesimal,* $a \geq 4$.

Theorem 1 will be proved in Section 4, while Theorem 2 will be proved in Section 5. We also announce here some further results, deferring their proofs to the supplement [5]:

Theorem 3. $\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}$, $a \geq 4,\ b \geq 4$.

Theorem 4. $\mathrm{T}^a\square\square\mathrm{F}^b = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, $a > b \geq 2$.

Theorem 5. $\mathrm{T}^a\square\square\square\mathrm{FFF} = a - \frac{7}{2}$, $a \geq 5$.

We conclude with conjectures based on empirical evidence from our computer program, we make 5 conjectures in Section 6, refining Erickson's conjecture E5 into two stronger conjectures.

2. Background

To be able to understand the present article, the readers need a minimum knowledge of combinatorial game theory, that can be found in [1]. In particular, readers should be familiar with the notions of *value* of a game and sum games.

We recall the notions of the sum and inequality of two games, the dominated options rule, and the bypass reversible move rule, from [1, pp. 33, 62-64].

2.1. Sum and Inequality of two games.

Let $G = \{G^L \mid G^R\}$, and let $H = \{H^L \mid H^R\}$. Then $G + H := \{G^L + H, G + H^L \mid G^R + H, G + H^R\}$.

$G \geq 0$ if Right goes first and Left wins.
$G \leq 0$ if Left goes first and Right wins.
$G \geq H \iff G + (-H) \geq 0$.

2.2. Dominated options rule.

Let $G = \{A, B, C, ... \mid D, E, F, ..\}$.
If $A \geq B$ and $D \geq E$ then $G = \{A, C, ... \mid E, F, ..\}$.

2.3. Bypassing right's reversible move rule.

$G = H$ if $D^L \geq G$ (see Figure 1).

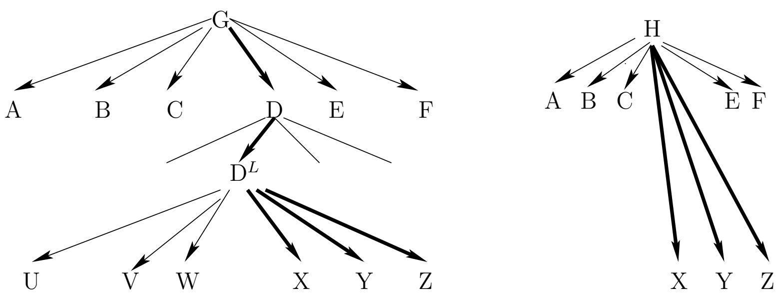

Figure 1. Bypass reversible move rule.

2.4. Special notation.

The only special notations we use are $*(= \{0 \mid 0\})$ and $n*(= \{n \mid n\})$. We will not use any shorthand notation like $\uparrow$, $\Uparrow$, etc.

3. Empirical Evidence

We present the values of some starting positions in this section. We have a fast program written in Java to calculate the outcome of the sum of two given positions (=,>,<,|| (not comparable)). This program does not calculate the value of the sum of two games. It only gives the outcome. It works well with the positions that have a simple value. The author's brother and the author wrote this program originally to check the value of the game of the form $\mathrm{T}^a\square^b\mathrm{F}^a$ for which so far the values of the game are 0 or * except the column b=2 which will be proved to be infinitesimal when $a \geq 4$. We present the tables here.

3.1. **Values for** $\mathrm{T}^a\square^b\mathrm{F}^a$**.**

$a\backslash b$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0	$*$	0
2	$*$	$*$	$*$	$*$	0	0	$*$	0	0	0	0	0	$*$	0	0	0	0	0	$*$	0
3	$*$	$\pm\frac{1}{8}$	0	$*$	0	$*$	0	0	0	0	0	0	0	$*$	0	0	0	0	0	0
4	$*$	N	$*$	0	0	0	0	$*$	$*$	0	0	0	0	0	0	0	0	0	0	0
5	$*$	N	$*$	$*$	$*$	0	$*$	0	0	0	$\neq 0$	0								
6	$*$	N	$*$	$*$	$*$	$*$	$*$	$\neq 0$												
7	$*$	N	$*$	$*$	$*$	$\neq 0$	$\neq 0$													
8	$*$	N	$*$	$*$	$*$															
9	$*$	N	$*$	$*$																
10	$*$	N	$*$	$*$																

Remarks:

(1) For b where $21 \leq b \leq 103$, $\mathrm{T}^2\square^b\mathrm{F}^2 = 0$ except $b = 25,31,37,43,49,55,61,67,73,79,85,91,97,103$.
(2) For b where $21 \leq b \leq 53$, $\mathrm{T}^3\square^b\mathrm{F}^3 = 0$ except $b = 29$.
(3) The notation N stands for an infinitesimal. In this table, they are long. We do not write them out.

3.2. **Values for** $\mathrm{T}^{a+1}\square^b\mathrm{F}^a$**.**

$a\backslash b$	3	4	5	6
1	$2*$	3	$4*$	5
2	$1*$	$2*$	3	$\frac{7}{2}$
3	$\{1 \mid \{\frac{1}{2} \mid 0\}\}$	$1*$	2	$\frac{11}{4}$
4	$1*$	$\{\{2* \mid 1*\} \mid\mid \{\frac{1}{2} \mid 0\}\}$	1	2
5	$1*$	$1*$	$\frac{5}{4} < V < 2$	1
6	$1*$	$1*$	$2*$	$1 < V < 2$
7	$1*$	$1*$	$\mid\mid 2$	
8	$1*$	$1*$	$\mid\mid 2$	

$a\backslash b$	7	8	9	10	11	12	13	14	15	16
1	$6*$	7	$8*$	9	$10*$	11	$12*$	13	$14*$	15
2	$5*$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{19}{2}$	$\frac{41}{4}$	$\frac{23}{2}$	$12*$	13
3	$\frac{15}{4}$	$\frac{9}{2}$	$< \frac{11}{2}$	< 6						
4	$\frac{5}{2} < V < 3$	< 4								
5	$\frac{5}{2} < V < 3$									
6	< 2 and $\mid\mid 3$									

3.3. Values for $T^{a+2}\square^{b}F^{a}$.

$a\backslash b$	3	4	5	6	7	8	9	10	11	12
1	$4*$	6	$8*$	10	$12*$	14	$16*$	18	$20*$	22
2	$2*$	$4*$	$\frac{95}{16}$	$\frac{15}{2}$	$9*$	$\{11 \mid 11*\}$	12	$\frac{29}{2}$	$15*$	$17*$
3	$\frac{3}{2}$	$\{\{\frac{5}{2} \mid 2\} \mid\mid 2\}$	4	$\{\frac{11}{2} \mid \frac{41}{8}\}$	7					
4	$2*$	$\{\{4* \mid 2*\} \mid\mid \{\frac{3}{2} \mid 1\}\}$	3	4	$5*$					
5	$2*$	$2*$	$3 < V < 4$	3	5					
6	$2*$	$2*$	$\mid\mid 4$							
7	$2*$	$2*$	< 3							

3.4. Values for $T^{a+3}\square^{b}F^{a}$.

$a\backslash b$	3	4	5	6	7	8
1	$6*$	9	$12*$	15	$18*$	21
2	$3*$	$\{6 \mid \frac{11}{2}\})$	$\{\frac{17}{2} \mid 8\})$	$11*$	13	$\frac{31}{2}$
3	$\frac{5}{2}$	L	$\frac{41}{8}$	$\mid\mid 8$		
4	$3*$	$\frac{5}{2}$	5	$\mid\mid 5$		
5	$3*$	$\mid\mid 3$	$5 < V < 6$			
6	$3*$	$3*$				

Remarks:

(1) In section 3.2, 3.3 and 3.4, we drop the values of the first two columns where $b = 1, 2$ since they are the results from theorem 1 and 4 we claimed in section 1 and $T^{a}\square\square F$ in [4] and [6].
(2) The value of $T^{6}\square^{4}F^{3}$ in section 3.4 is long. We do not write it out.

Corollary 1. *Conjecture E4 is false.*

Proof: From the table, $T^{7}\square^{7}F^{6} > 2$. $\square$

The author believes there are no patterns in a for positions of the form $T^{a+k}\square^{a+l}F^{a}$ for any fixed $k \geq 1,\ l \geq 0$.

4. Positions with one $\square$

In this section we classify all the positions that have one $\square$. We can compute the values of all positions in this class by 8 simple lemmas. All lemmas could be easily proved by induction. We omit all the proofs here.

Notations
$O(x) = \{0 \mid x\}$.
$O^{a}(x) = O(...(O(O(x))))$ a times.
$\widetilde{\mathrm{L}}$ = empty or any combination of T and F that has F as its right most entry. For example TTFTF.
$\widetilde{\mathrm{R}}$ = empty or any combination of T and F that has T as its left most entry. For example

TFTTF.

Lemma 4.1. $\widetilde{\mathrm{L}}\mathrm{T}^a\square = a,\ a \geq 0$ *and* $P_1\mathrm{FF}\square P_2 = \square P_2$ *for any position* P_1 *and* P_2.

Lemma 4.2. *Death Leaps Principle(DLP): The position with one empty square for which the only possible move for both sides is a jump has value 0.*

Example $\mathrm{TFTTF}\square\mathrm{TFTFF} = 0$.

Lemma 4.3. $\widetilde{\mathrm{L}}\mathrm{T}\square\mathrm{F}\widetilde{\mathrm{R}} = *$.

Lemma 4.4. $\widetilde{\mathrm{L}}\mathrm{T}^a\square\mathrm{F}^b\widetilde{\mathrm{R}} = *,\quad a \geq 2, b \geq 2.$

Lemma 4.5. $\widetilde{\mathrm{L}}\mathrm{T}^a\square(\mathrm{TF})^b = \{a-1 \mid (\frac{1}{2})^{b-1}\},\quad a \geq 1, b \geq 1.$

Lemma 4.6. $\widetilde{\mathrm{L}}\mathrm{T}^a\square\mathrm{F}(\mathrm{TF})^b = \{\{a-2 \mid (\frac{1}{2})^b\} \mid\mid 0\},\quad a \geq 2, b \geq 0.$

Lemma 4.7. $\widetilde{\mathrm{L}}\mathrm{T}^a\square(\mathrm{TF})^b\mathrm{TF}^c\widetilde{\mathrm{R}} = \{a-1 \mid (O^b(\widetilde{\mathrm{L}}\mathrm{T}^a\mathrm{F}(\mathrm{TF})^b\mathrm{T}\square\mathrm{F}^{c-1}\widetilde{\mathrm{R}})\},\quad a \geq 1, b \geq 0, c \geq 2.$

Lemma 4.8. $\widetilde{\mathrm{L}}\mathrm{T}^a\square\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^c\widetilde{\mathrm{R}} = \{\{a-2 \mid O^{b+1}(\widetilde{\mathrm{L}}\mathrm{T}^{a-1}\mathrm{F}(\mathrm{TF})^{b+1}\mathrm{T}\square\mathrm{F}^{c-1}\widetilde{\mathrm{R}})\} \mid\mid 0\},\quad a \geq 2, b \geq 0, c \geq 2.$

Example $\mathrm{T}^a\square\mathrm{F}(\mathrm{TF})^b\mathrm{TF}^2 = \{\{a-2 \mid O^{b+1}(*)\} \mid\mid 0\},\quad a \geq 2, b \geq 0.$

Remark: The recurrences in lemma 4.7 and lemma 4.8 will finally simplify to the known positions from the previous lemmas.

5. Positions $\mathrm{T}^a\square\square\mathrm{F}^a$

5.1. **Convention.** We explain here the notation we use in section 5.2.

For $G \leq H$ or equivalently $G - H \leq 0$, we show Right can win when Left moves first in the game $G - H$. We show that for each of the possible Left choices, Right finds a response that makes him win the game

Example: To show: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{FT}\square\mathrm{F}^b \leq \frac{1}{2};\quad k \geq 0, a \geq 0, b \geq 1.$

$$\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\mathrm{T}^k\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \leq \overset{\overset{3}{\leftarrow}}{\tfrac{1}{2}}\ .$$

Left has three choices to play, moving T's on the left hand side or making a move on the right hand side. Right could response to each of the Left's move as following:

First choice: Right responses by moving in the left option of $\{0 \mid 1\}$ on the right hand side. This leads to position in case 1 below:

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \leq 0$.

Second choice: Right responses by moving the left most F. This leads to position in case 2 below:

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\mathrm{FT}\square\mathrm{F}^{b} \leq \frac{1}{2}$.

Third choice: Left picks the right option of $\{0 \mid 1\}$ on the right hand side. Right responses by moving the right most F. This leads to position in case 3 below:

Case 3: $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\mathrm{FTF}\square\mathrm{F}^{b-1} \leq 1$.

5.2. **Proof of Theorem 2.** In this section, we show $\mathrm{T}^{a}\square\square\mathrm{F}^{a}$ is an infinitesimal for $a \geq 4$. The observation comes from table 3.1 when $b = 2$. We first state the following two lemmas. The proof of the first lemma is similar to the proof of DLP. We left it to the reader.

Lemma 5.1. ***One side Death Leap Principle*** *(One side DLP): if X is a position where there is no two or more consecutive empty square and the only possible move of Left is a jump then $X \leq 0$.*

Examples: $\mathrm{TTF}\square\mathrm{TTF}\square\mathrm{F} \leq 0$ and $\mathrm{TTTF}\square\mathrm{F}\square\mathrm{TF} \leq 0$.

Lemma 5.2. *For any fixed integer $n \geq 3$, $\widetilde{\mathrm{L}}\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\mathrm{FT}\square\mathrm{F}^{b} \leq \frac{1}{2^n}$, $k \geq 0, a \geq 0, b \geq 1$.*

Prove: By induction on a.

Base Case: $a = 0$, $\square\mathrm{T}^{k}\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{b} \overset{\overset{2}{\leftarrow}}{\leq} \frac{1}{2^n}$.

Case 1: $\square\mathrm{T}^{k}\mathrm{F}\square\mathrm{TF}^{b} \leq 0$, true by one side DLP.

Case 2: $\square\mathrm{T}^{k}\mathrm{FTF}\square\mathrm{F}^{b-1} \leq \frac{2}{2^n}$. The left hand side is ≤ 0 by one side DLP.

Induction Step: $\widetilde{\mathrm{L}}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a}}\ \mathrm{F}\square\mathrm{T}^{k}\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{b} \overset{\overset{3}{\leftarrow}}{\leq} \frac{1}{2^n}$.

Case 1: $\widetilde{\mathrm{L}}\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\mathrm{F}\square\mathrm{TF}^{b} \leq 0$, true by one side DLP.

Case 2: $\widetilde{\mathrm{L}}\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\mathrm{FT}\square\mathrm{F}^{b} \leq \frac{1}{2^n}$, true by induction.

Case 3: $\widetilde{\mathrm{L}}\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\mathrm{FTF}\square\mathrm{F}^{b-1} \leq \frac{2}{2^n}$. The left hand side is ≤ 0 by one side DLP.

We are now ready to prove the main theorem.

Theorem 6. $\mathrm{T}^{a}\square\square\mathrm{F}^{a}$ *is an infinitesimal* , $a \geq 4$.

By symmetry we only need to show: For any fixed integer $n \geq 3$, $\mathrm{T}^{a}\square\square\mathrm{F}^{a} \leq \frac{1}{2^n}$, $a \geq 4$.

$\overset{\overset{I}{\rightarrow}}{\mathrm{T}^{a}}\ \square\square\mathrm{F}^{a} \overset{\overset{II}{\leftarrow}}{\leq} \frac{1}{2^n}$.

I) $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}^{a-1} \overset{\overset{3}{\leftarrow}}{\leq} \frac{1}{2^n}$

II) $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}}\ \square\mathrm{F}\square\mathrm{F}^{a-1} \overset{\overset{2}{\leftarrow}}{\leq} \frac{2}{2^n}$

I) Case 1: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{F}\square\mathrm{TF}^{a-1} \leq \overset{\overset{2}{\leftarrow}}{\frac{1}{2^n}}$

Case 1.1: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\square\mathrm{TF}^{a-1} \leq \overset{\overset{2}{\leftarrow}}{\frac{1}{2^n}}$

Case 1.1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{a-2} \leq \frac{1}{2^n}$, true by lemma 5.2.

Case 1.1.2: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{a-2} \leq \overset{\overset{3}{\leftarrow}}{\frac{2}{2^n}}$

Case 1.1.2.1: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\square\mathrm{TF}^{a-2} \leq \overset{\overset{2}{\leftarrow}}{\frac{2}{2^n}}$

Case 1.1.2.1.1: $\square\mathrm{T}\square\mathrm{TF}^{a-2} \leq \frac{2}{2^n}$.

The value of the left hand side is $\{0 \mid\mid \{0 \mid \{-1 \mid 5-a\}\}\}$ (see ClassA22 in [6]) which confirms the case above.

Case 1.1.2.1.2: $\mathrm{T}^{a-2}\mathrm{FTF}\square\mathrm{FT}\square\mathrm{F}^{a-3} \leq \frac{4}{2^n}$, true by lemma 5.2.

Case 1.1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFTF}\square\mathrm{F}^{a-3} \leq \frac{2}{2^n}$, true by one side DLP.

Case 1.1.2.3: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}^{a-3} \leq \overset{\overset{3}{\leftarrow}}{\frac{4}{2^n}}$

Case 1.1.2.3.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{F}\square\mathrm{FTF}^{a-3} \leq 0$

$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFF}\square\mathrm{TF}^{a-3} \leq 0$, true by one side DLP.

Case 1.1.2.3.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFTFF}\square\mathrm{F}^{a-4} \leq \frac{4}{2^n}$, true by one side DLP.

Case 1.1.2.3.3: $\mathrm{T}^{a-2}\mathrm{FTF}\square\mathrm{TF}\square\mathrm{F}^{a-3} \leq \frac{8}{2^n}$, true by one side DLP.

Case 1.2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{FF}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^{a-2} \leq \overset{\overset{3}{\leftarrow}}{\frac{2}{2^n}}$

Case 1.2.1: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{F}\square\mathrm{TF}^{a-2} \leq \frac{2}{2^n}$, true by one side DLP.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{FT}\square\mathrm{F}^{a-2} \leq \frac{2}{2^n}$. This is the case 1.1.2

Case 1.2.3: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{FT}\square\mathrm{F}^{a-2} \leq \frac{4}{2^n}$, true by lemma 5.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TTFF}\square\mathrm{F}^{a-2} \leq \frac{1}{2^n}$. The left hand side is 0.

Case 3: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{TFF}\square\mathrm{F}^{a-2} \leq \overset{\overset{2}{\leftarrow}}{\frac{2}{2^n}}$

Case 3.1: $\mathrm{T}^{a-2}\square\mathrm{TTFF}\square\mathrm{F}^{a-2} \leq 0$. The left hand side is clearly 0.

Case 3.2: $\mathrm{T}^{a-1}\mathrm{FT}\square\mathrm{F}\square\mathrm{F}^{a-2} \leq \frac{4}{2^n}$. The left hand side is ≤ 0.

Since the only possible left move leads to $\mathrm{T}^{a-1}\mathrm{FFT}\Box\Box\mathrm{F}^{a-2} = 3 - a$.

II) Case 1: $\mathrm{T}^{a-1}\Box\mathrm{TFF}\Box\mathrm{F}^{a-2} \leq \frac{2}{2^n}$. This is I) case 3.

Case2: $\overset{\xrightarrow{1}}{\mathrm{T}^a}\Box\mathrm{FF}\Box\mathrm{F}^{a-2} \overset{\xleftarrow{2}}{\leq} \frac{4}{2^n}$

Case 2.1: $\mathrm{T}^{a-1}\mathrm{FT}\Box\mathrm{F}\Box\mathrm{F}^{a-2} \leq \frac{4}{2^n}$. This is I) case 3.2.

Case 2.2: $\mathrm{T}^{a}\mathrm{F}\Box\mathrm{F}\Box\mathrm{F}^{a-2} \leq \frac{8}{2^n}$, true by one side DLP. $\Box$

6. New Conjectures and Future Work

We believe there are still a lot of nice patterns and conjectures in this game that we overlooked. We will have more information when RAM gets cheaper and Maple gets faster. We make the following conjectures.

Conjectures:

T1: Assume $a \geq 1$, $b \geq 0$, $L \geq 0$ and $R \geq 0$

$$\text{T1.1: } \Box^R\mathrm{T}^a\Box^b\mathrm{F}\Box^R = \begin{cases} \{\{a-2 \mid 1\} \mid 0\} & \text{if } R = 0 \text{ and } b = 1 \\ (a-1)(b-1+R) & \text{if } b \text{ is even and } (R,b) \neq (0,0) \\ (a-1)(b-1+R)* & \text{if } b \text{ is odd and } (R,b) \neq (0,1) \end{cases}$$

$$\text{T1.2: For } R \geq 1, \quad \Box^{R-1}\mathrm{T}^a\Box^b\mathrm{F}\Box^R = \begin{cases} (a-1)(b-1+R) & \text{if } b \text{ is even} \\ 1/2 + (a-1)(b-1+R) & \text{if } b \text{ is odd} \end{cases}$$

T1.3: For $R - L \geq 2$, $\Box^L\mathrm{T}^a\Box^b\mathrm{F}\Box^R = (R-L-1) + (a-1)(b-1+R)$

$$\text{T2: For } a \geq 7,\ \mathrm{TT}\Box^a\mathrm{FF} = \begin{cases} * & \text{when } a = 7 + 6n, \quad n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

T3: $\mathrm{T}^a\Box^b\mathrm{F}^a = *$ for any $a > b > 0$, except for $b = 2$.
T4: $\mathrm{T}^a\Box^b\mathrm{F}^a = 0$ or $*$ for any $b \geq a > 0$.
T5: For any fixed integer $C \geq 3$, $\exists a_0$ such that $\mathrm{T}^C\Box^a\mathrm{F}^C = 0$ for all $a \geq a_0$.

Future Work:

1) Categorize all the positions that have exactly one F (general class B1) (conjecture T1 might be a good start).

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